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The system of generalized vector equilibrium problems with applications

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Abstract. In this paper, we introduce the system of generalized vector equilibrium problems which includes as special cases the system of generalized implicit vector variational inequality problems, the system of generalized vector variational and variational-like inequality problems and the system of vector equilibrium problems. By using a maximal element theorem, we establish existence results for a solution of these systems. As an application, we derive existence results for a solution of a more general Nash equilibrium problem for vector-valued functions.

Key words: Vector equilibrium problem, Vector variational inequality, Vector optimization problem, Nash equilibrium problem, Maximal element theorem

1. Introduction

Recently the generalized vector equilibrium problem (in short, GVEP) has been studied in (Ansari et al., 1997, 2001; Oettli 1997; Oettli and Schläger, 1998a, b; Ansari and Yao, 1999a; Konnov and Yao, 1999; Song, 2000). It includes as special cases generalized implicit vector variational inequality problems, generalized vector variational and variational-like inequality problems and vector equilibrium problems. For further details on generalized vector variational and variational-like inequality problems and vector equilibrium problems, we refer to (Lee et al., 1996b; Bianchi et al., 1997; Oettli, 1997; Hadjisavvas and Schaible, 1998a,b; Tan and Tinh, 1998; Giannessi, 2000) and references therein.

Very recently, the system of vector equilibrium problems (in short, SVEP), a family of equilibrium problems for vector-valued bifunctions defined on a product set, was introduced in (Ansari et al., 2000). The (SVEP) contains as special cases the system of equilibrium problems, the system of variational inequality problems (Ansari and Yao, 1999b), the system of vector variational inequality problems, the system of vector optimization problems and the Nash equilibrium problem for vector-valued functions.

In this paper, we introduce the system of generalized vector equilibrium problems (in short, SGVEP) which contains the system of generalized implicit vector variational inequality problems, the system of generalized vector variational and variational-like inequality problems and (SVEP). We establish some existence results for a solution of the (SGVEP) using a special case of a maximal element theorem for a family of multivalued maps due to Deguire et al. (1999). We also derive some existence results for a solution of the system of generalized implicit vector variational inequality problems, the system of generalized vector variational and variational-like inequality problems and (SVEP). As an application, we give some existence results for a solution of a more general Nash equilibrium problem for vector-valued functions.

2. Formulations and preliminaries

Let I be an index set, and for each $i \in I$, let X_i be a Hausdorff topological vector space. Consider a family of nonempty convex subsets $\{K_i\}_{i\in I}$ with K_i in X_i. Throughout this paper, $K = \prod_{i\in I} K_i$ and $X = \prod_{i\in I} X_i$. For each $i \in I$, let Y_i be a topological vector space and let $C_i : K \to \Pi(Y_i)$ be a multivalued map such that for each $x \in K$, $C_i(x)$ is a proper, closed and convex cone with int $C_i(x) \neq \emptyset$, where int C_i and $\Pi(Y_i)$ denote the interior of C_i and the family of all nonempty subsets of Y_i, respectively. For each $i \in I$, let $F_i : K \times K_i \to \Pi(Y_i)$ be a multivalued bifunction. We consider the following *system of generalized vector equilibrium problems* (in short, SGVEP) which is to find $\bar{x} \in K$ such that for each $i \in I$,

$$F_i(\bar{x}, y_i) \nsubseteq -\text{int}C_i(\bar{x}), \quad \text{for all } \; y_i \in K_i.$$

If the index set I is a singleton, then the (SGVEP) reduces to a generalized vector equilibrium problem studied in (Oettli and Schläger, 1998b; Ansari and Yao, 1999a; Konnov and Yao, 1999; Ansari et al., 2001) which contains generalized implicit vector variational inequality problems (Lee and Kum, 2000), generalized vector variational and variational-like inequality problems and vector equilibrium problems as special cases; see for example (Lee et al., 1996b; Bianchi et al., 1997; Oettli, 1997; Hadjisavvas and Schaible, 1998a,b; Tan and Tinh, 1998; Giannessi, 2000) and references therein.

EXAMPLES OF (SGVEP):

For each $i \in I$, we denote by $L(X_i, Y_i)$ the space of the continuous linear operators from X_i into Y_i. Let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, let $T_i : K \to \Pi(D_i)$ be a multivalued map.

(1) *The System of Generalized Implicit Vector Variational Inequality Problems* (in short, SGIVVIP):
For each $i \in I$, let $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. The (SGIVVIP) is to find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall y_i \in K_i, \;\; \exists \bar{u}_i \in T_i(\bar{x}) \;\; : \;\; \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int}C_i(\bar{x}).$$

Setting for each $i \in I$,

$$F_i(x, y_i) = \psi_i(T_i(x), x_i, y_i) = \{\psi_i(u_i, x_i, y_i) : u_i \in T_i(x)\},$$

the (SGVEP) coincides with the (SGIVVIP).
For $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_-$ for all $x \in K$ and for each $i \in I$, the (SGIVVIP) is called *the system of generalized implicit variational inequality problems* considered and studied in (Ansari and Yao, 2000b).
If the index set I is a singleton, then the (SGIVVIP) reduces to the *generalized implicit vector variational inequality problem* considered and studied in (Lee and Kum, 2000).
The (SGIVVIP) contains the following problems as special cases:

(i) For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be a bifunction. If for each $i \in I$,

$$\psi_i(T_i(x), x_i, y_i) = \{\langle u_i, \eta_i(y_i, x_i)\rangle : u_i \in T_i(x)\},$$

then the (SGIVVIP) reduces to the *system of generalized vector variational-like inequality problems* (in short, SGVVLIP) which is to find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall y_i \in K_i, \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i)\rangle \notin -\text{int}C_i(\bar{x}),$$

where $\langle s_i, x_i\rangle$ denotes the evaluation of $s_i \in L(X_i, Y_i)$ at $x_i \in X_i$.
For $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_+$ for all $x \in K$ and for each $i \in I$, this problem is studied in (Ansari and Yao, 2000b) with application to the Nash equilibrium problem (Nash, 1951) for nonconvex and nondifferentiable functions.
In the case where the index set I is a singleton, this problem is considered and studied in (Giannessi, 2000) and references therein where it is used as a tool to prove the existence of a solution to vector optimization problems for nondifferentiable and nonconvex functions; see for example (Lee et al., 1996a; Ansari and Yao, 2000a) and the references therein.

(ii) If for each $i \in I$,

$$\psi_i(T_i(x), x_i, y_i) = \langle T_i(x), y_i - x_i\rangle = \{\langle u_i, y_i - x_i\rangle : u_i \in T_i(x)\},$$

then the (SGIVVIP) reduces to the *system of generalized vector variational inequality problems* (in short, SGVVIP) which is to find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall y_i \in K_i, \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \bar{u}_i, y_i - \bar{x}_i\rangle \notin -\text{int}C_i(\bar{x}).$$

If the index set I is a singleton, then the (SGVVIP) is studied in (Daniilidis and Hadjisavvas, 1996; Konnov and Yao, 1997; Hadjisavvas and Schaible, 1998b; Giannessi, 2000) and references therein with further applications in vector optimization theory. When $C_i(x) = \mathbb{R}_+$ and $Y_i = \mathbb{R}$, for each

$i \in I$ and for all $x \in K$, the (SGVVIP) is studied by Ansari and Yao (2000b) and Deguire et al. (1999), and used as a tool to find the solution of the Nash equilibrium problem for convex but nondifferentiable functions.

(2) *The System of Vector Equilibrium Problems* (in short, SVEP): for each $i \in I$, let F_i be a single-valued map. Then the (SGVEP) is equivalent to the following *system of vector equilibrium problems* (in short, SVEP), introduced and studied in (Ansari et al., 2000) with applications to the system of vector variational inequality problems and to the system of vector optimization problems which includes the Nash equilibrium problem for vector-valued functions as a special case:

$$\text{(SVEP)} \quad \begin{cases} \text{find } \bar{x} \in K \quad \text{such that for each } i \in I, \\ F_i(\bar{x}, y_i) \notin -\text{int}C_i(\bar{x}) \quad \text{for all } y_i \in K_i. \end{cases}$$

The (SVEP) contains the system of vector variational inequality problems as a special case, which describes a variety of important equilibrium models; for details see (Ansari et al., 2000) and the references therein.

Further, consider the following problem. For each $i \in I$, let $\varphi_i : K \to Y_i$ be a vector-valued function and let $K^i = \prod_{j \in I,\ j \neq i} K_j$ and we write $K = K^i \times K_i$. For $x \in K$, x^i denotes the projection of x onto K^i and hence we can write $x = (x^i, x_i)$. If for each $i \in I$,

$$F_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x),$$

then the (SVEP) is equivalent to the following Nash equilibrium problem for vector-valued functions: find $\bar{x} \in K$ such that for each $i \in I$,

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int}C_i(\bar{x}) \quad \text{for all } y_i \in K_i.$$

It is considered and studied by Tan and Tinh (1998), but for a finite index set I and $C_i(x) = C_i$ for all $x \in K$. For a finite index set I and $C_i(x) = C(x)$ for each $i \in I$, this problem is also analyzed in (Lee et al., 1996b).

Now we mention some definitions and a result which will be used in the sequel.

DEFINITION 1. (Luc, 1989; Bianchi et al., 1997) Let W be a topological vector space and Z be another topological vector space with a proper, closed and convex cone P. Let M be a nonempty convex subset of W. A function $\phi : M \to Z$ is called *P-quasiconvex* if, for all $\alpha \in Z$, the set $\{x \in M : \phi(x) - \alpha \in -P\}$ is convex.

REMARK 1. If ϕ is P-quasiconvex, then the set $\{x \in M : \phi(x) - \alpha \in -\text{int } P\}$ is also convex.

DEFINITION 2. (Berge, 1963) A multivalued map $T : W \to \Pi(Z)$ is called *upper semi-continuous on W* if T has compact values and for each $x_0 \in W$ and for any open set V in Z containing $T(x_0)$ there exists an open neighborhood U of x_0 in W such that $T(x) \subseteq V$ for all $x \in U$.

In the next section we shall use the following particular form of a maximal element theorem for a family of multivalued maps due to Deguire et al. (1999, Theorem 7).

Let W and Z be topological vector spaces. We recall that a point $\bar{x} \in W$ is said to be a *maximal element* of a multivalued map $F : W \to \Pi(Z) \cup \{\emptyset\}$ if $F(\bar{x}) = \emptyset$.

THEOREM 1. *(*Deguire et al., 1999*) Let $\{K_i\}_{i \in I}$ be a family of nonempty convex subsets where each K_i is contained in a Hausdorff topological vector space X_i. For each $i \in I$, let $S_i : K \to \Pi(K_i) \cup \{\emptyset\}$ be a multivalued map such that*

(i) *for each $x \in K$, $S_i(x)$ is convex,*

(ii) *for each $x \in K$, $x_i \notin S_i(x)$, where x_i is the ith component of x,*

(iii) *for each $y_i \in K_i$, $S_i^{-1}(y_i)$ is open in K.*

Suppose that there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exists $i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$. Then there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

3. Existence results

In this section, we first establish some existence results for a solution of the (SGVEP). Then we derive existence results for a solution of the (SGIVVIP), (SGVVLIP), (SGVVIP) and (SVEP).

DEFINITION 3. (Ansari and Yao, 1999a) Let W and Z be topological vector spaces and M a nonempty convex subset of W and let $P : M \to \Pi(Z)$ be a multivalued map such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. A multivalued bifunction $F : M \times M \to \Pi(Z) \cup \{\emptyset\}$ is called *$P(x)$-quasiconvex-like* if for all $x, y_1, y_2 \in M$ and $t \in [0, 1]$, we have either

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_1) - P(x),$$

or

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_2) - P(x).$$

THEOREM 2. *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i and let Y_i be a topological vector space. For each $i \in I$, let $F_i : K \times K_i \to \Pi(Y_i)$ be a multivalued bifunction. For each $i \in I$, assume that*

(i) *$C_i : K \to \Pi(Y_i)$ is a multivalued map such that for each $x \in K$, $C_i(x)$ is a proper, closed and convex cone in Y_i with* $\mathrm{int}C_i(x) \neq \emptyset$;

(ii) *for all $x \in K$, $F_i(x, x_i) \nsubseteq -\mathrm{int}C_i(x)$, where x_i is the ith component of x;*

(iii) *for all* $x \in K$*, the set* $\{y_i \in K_i : F_i(x, y_i) \subseteq -\mathrm{int}C_i(x)\}$ *is convex;*
(iv) *for all* $y_i \in K_i$*, the set* $\{x \in K : F_i(x, y_i) \not\subseteq -\mathrm{int}C_i(x)\}$ *is closed in* K*;*
(v) *there exist a nonempty compact subset* N *of* K *and a nonempty compact convex subset* B_i *of* K_i *for each* $i \in I$ *such that for each* $x \in K \setminus N$ *there exist* $i \in I$ *and* $\tilde{y}_i \in B_i$ *such that* $F_i(x, \tilde{y}_i) \subseteq -\mathrm{int}C_i(x)$.

Then the (SGVEP) has a solution.

Proof. For each $i \in I$, we define a multivalued map $S_i : K \to \Pi(K_i) \cup \{\emptyset\}$ by

$$S_i(x) = \{y_i \in K_i : F_i(x, y_i) \subseteq -\mathrm{int}C_i(x)\} \quad \text{for all } x \in K.$$

By condition (iii) for each $i \in I$ and for all $x \in K$, $S_i(x)$ is convex. Condition (ii) implies that for all $x \in K$, $x_i \notin S_i(x)$.

By condition (iv) for each $i \in I$ and for all $y_i \in K_i$, $S_i^{-1}(y_i)$ is open in K. By (v), for each $x \in K \setminus N$ there exists $i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$. Thus all conditions of Theorem 1 are satisfied, and hence there exists $\bar{x} \in N$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$, that is, for each $i \in I$,

$$F_i(\bar{x}, y_i) \not\subseteq -\mathrm{int}C_i(\bar{x}), \quad \text{for all } y_i \in K_i.$$

Hence the result follows.

THEOREM 3. *For each* $i \in I$*, let* K_i*,* X_i*,* Y_i*,* C_i *and* F_i *be the same as in Theorem 2. For each* $i \in I$*, assume that*
(i) $Q_i : K \to \Pi(Y_i)$ *is a multivalued map defined as* $Q_i(x) = Y_i \setminus \{-\mathrm{int}C_i(x)\}$ *for all* $x \in K$ *such that its graph is closed in* $K \times Y_i$*;*
(ii) *for all* $x \in K$*,* $F_i(x, x_i) \not\subseteq -\mathrm{int}C_i(x)$*, where* x_i *is the* i*th component of* x*;*
(iii) F_i *is* $C_i(x)$*-quasiconvex-like;*
(iv) *for all* $y_i \in K_i$*, the multivalued map* $x \mapsto F_i(x, y_i)$ *is upper semicontinuous on* K*;*
(v) *there exist a nonempty compact subset* N *of* K *and a nonempty compact convex subset* B_i *of* K_i *for each* $i \in I$ *such that for each* $x \in K \setminus N$ *there exist* $i \in I$ *and* $\tilde{y}_i \in B_i$ *such that* $F_i(x, \tilde{y}_i) \subseteq -\mathrm{int}C_i(x)$.

Then the (SGVEP) has a solution.

Proof. For each $i \in I$, let S_i be the same as in the proof of Theorem 2. Then by (iii), for each $i \in I$ and for all $x \in K$, $S_i(x)$ is convex (see for example the proof of Theorem 2.1 in Ansari and Yao (1999a)).

By using conditions (i) and (iv), from the proof of Theorem 2.1 in Ansari and Yao (1999a), we see that the set $\{x \in K : F_i(x, y_i) \not\subseteq -\mathrm{int}C_i(x)\}$ is closed in K. Then the result follows from the proof of Theorem 2.

DEFINITION 4. (Ansari et al., 2001) Let W and Z be topological vector spaces, M a nonempty convex subset of W and D a nonempty subset of $L(W, Z)$. Let $T : M \to \Pi(D)$ and $P : M \to \Pi(Z)$ be multivalued maps such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. A function ψ :

$D \times M \times M \to Z$ is called $P(x)$-*quasiconvex-like* if for all $x, y_1, y_2 \in M$ and $t \in [0, 1]$, we have either for all $u \in T(x)$,

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_1) - P(x),$$

or

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_2) - P(x).$$

By using Theorems 2 and 3, we derive the following existence result for a solution of the (SGIVVIP).

COROLLARY 1. *For each $i \in I$, let K_i, X_i, Y_i, C_i and Q_i be the same as in Theorem 3 and let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, $T_i : K \to \Pi(D_i)$ be an upper semicontinuous multivalued map and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. For each $i \in I$, assume that*

(i) *for all $x \in K$ and $u_i \in T_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\text{int}C_i(x)$, where x_i is the ith component of x;*
(ii) *ψ_i is $C_i(x)$-quasiconvex-like;*
(iii) *for all $y_i \in K_i$, the map $(u_i, x_i) \mapsto \psi_i(u_i, x_i, y_i)$ is upper semicontinuous on $D_i \times K_i$;*
(iv) *there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ such that $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int}C_i(x)$ for all $u_i \in T_i(x)$.*

Then the (SGIVVIP) has a solution.

Proof. For each $i \in I$, we set

$$F_i(x, y_i) = \psi_i(T_i(x), x_i, y_i) = \{\psi_i(u_i, x_i, y_i) : u_i \in T_i(x)\}$$

for all $x \in K$ and $y_i \in K_i$. Then, all the conditions of Theorem 2 can easily be verified except for condition (iv). Hence we only need to prove that the set

$$A = \{x \in K : \exists u_i \in T_i(x) \text{ s.t. } \psi_i(u_i, x_i, y_i) \notin -\text{int}C_i(x)\}$$

is closed in K for all $y_i \in K_i$. We prove it for a fixed i.

Let $\{x_\lambda\}$ be a net in A such that x_λ converges to $x^* \in K$. Then

$$\exists u_{i_\lambda} \in T_i(x_\lambda) \text{ s.t. } \psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \notin -\text{int}C_i(x_\lambda),$$

where x_{i_λ} is the ith component of x_λ, and therefore

$$\psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \in Q_i(x_\lambda).$$

Let $\mathcal{F} = \{x_\lambda\} \cup \{x^*\}$. Then $\mathcal{F}$ is compact and $u_{i_\lambda} \in T_i(\mathcal{F})$ which is compact. Therefore u_{i_λ} has a convergent subnet with limit u_{i_*}. Without loss of generality, we may assume that $\{u_{i_\lambda}\}$ converges to u_{i_*}. Then by upper semicontinuity of T, we have $u_{i_*} \in T_i(x^*)$. Since $\psi_i(.,.,y_i)$ is continuous and the graph of Q_i is closed, we have

$$\psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \text{ converges to } \psi_i(u_{i_*}, x_{i_*}, y_i) \in Q_i(x^*),$$

and hence $\psi_i(u_{i_*}, x_{i_*}, y_i) \notin -\text{int}C_i(x^*)$. Therefore, $x^* \in A$ and thus A is closed in K. This completes the proof.

REMARK 2. Corollary 1 strengthens Theorem 3.2 in Lee and Kum (2000) in several ways since our assumptions are weaker.

Let W and Z be Hausdorff topological vector spaces and σ be the family of all bounded subsets of W whose union is total in W, that is, the linear hull of $\bigcup\{U : U \in \sigma\}$ is dense in W. Let $\mathcal{B}$ be a neighborhood base of 0 in Z. When U runs through σ, V through $\mathcal{B}$, the family

$$M(U, V) = \{\xi \in L(W, Z) : \cup_{x\in U}\langle \xi, x\rangle \subseteq V\}$$

is a neighborhood base of 0 in $L(W, Z)$ for a unique translation-invariant topology, called the *topology of uniform convergence* on the sets $U \in \sigma$, or, briefly, the σ-topology (see Schaefer (1971), pp. 79–80).

In order to derive existence results for solutions of the (SGVVLIP) and (SGVVIP) from Corollary 1, we need the following useful result due to Ding and Tarafdar (2000).

LEMMA 1. *Let W and Z be real Hausdorff topological vector spaces and $L(W, Z)$ be the topological vector space under the σ-topology. Then, the bilinear mapping $\langle ., .\rangle : L(W, Z) \times W \to Z$ is continuous on $L(W, Z) \times W$.*

COROLLARY 2. *For each $i \in I$, let X_i and Y_i be Hausdorff topological vector spaces and let K_i, C_i, Q_i, D_i and T_i be the same as in Corollary 1. For each $i \in I$, let $L(X_i, Y_i)$ be equipped with the σ-topology. For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be affine in the first argument and continuous in the second argument such that $\eta_i(x_i, x_i) = 0$ for all $x_i \in K_i$. Assume that there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ such that $\langle u_i, \eta_i(\tilde{y}_i, x_i)\rangle \in -\text{int}C_i(x)$ for all $u_i \in T_i(x)$. Then the (SGVVLIP) has a solution.*

COROLLARY 3. *For each $i \in I$, let K_i, X_i, Y_i, C_i, Q_i, D_i and T_i be the same as in Corollary 2 and let $L(X_i, Y_i)$ be equipped with the σ-topology. Assume that there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ such that $\langle u_i, \tilde{y}_i - x_i\rangle \in -\text{int}C_i(x)$ for all $u_i \in T_i(x)$. Then the (SGVVIP) has a solution.*

DEFINITION 5. (Ansari and Yao, 1999a) Let W and Z be topological vector spaces and M a nonempty convex subset of W and let $P : M \to \Pi(Z)$ be a multivalued map such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. A bifunction $f : M \times M \to Z$ is called *$P(x)$-quasiconvex-like* if for all $x, y_1, y_2 \in M$ and $t \in [0, 1]$, we have either

$$f(x, ty_1 + (1-t)y_2) \in f(x, y_1) - P(x),$$

or

$$f(x, ty_1 + (1-t)y_2) \in f(x, y_2) - P(x).$$

From Theorem 2, we derive the following existence result for a solution of the (SVEP).

COROLLARY 4. *For each $i \in I$, let K_i, X_i, Y_i, C_i and Q_i be the same as in Theorem 3. For each $i \in I$, let $f_i : K \times K_i \to Y_i$ be a bifunction. For each $i \in I$, assume that*

(i) *for all $x \in K$, $f_i(x, x_i) \notin -\mathrm{int}C_i(x)$ where x_i is the ith component of x;*

(ii) *f_i is $C_i(x)$-quasiconvex-like;*

(iii) *for all $y_i \in K_i$, the map $x \mapsto f_i(x, y_i)$ is continuous on K,*

(iv) *there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ such that $f_i(x, \tilde{y}_i) \in -\mathrm{int}C_i(x)$.*

Then the (SVEP) has a solution.

REMARK 3. Corollaries 1, 2 and 3 improve Theorem 2.2 and Corollaries 2.2 and 2.3 in Ansari and Yao (2000b), respectively, and extend them to vector-valued functions.

4. An application

Throughout this section, unless otherwise specified, we assume a finite index set $I = \{1, \ldots, n\}$. For each $i \in I$, X_i and Y_i are finite dimensional Euclidean spaces $\mathbb{R}^{p_i}$ and $\mathbb{R}^{q_i}$, respectively. Let $\{K_i\}_{i \in I}$ be a family of nonempty convex subsets with each K_i in X_i. For each $i \in I$, let $C_i : K \to \Pi(Y_i)$ be a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper, closed and convex cone with $\mathrm{int}C_i(x) \neq \emptyset$ and $\mathbb{R}^{q_i}_+ \subseteq C_i(x)$. For each $i \in I$, let $\varphi_i : K \to Y_i$ be a given vector-valued function. We consider the following *system of vector optimization problems* (in short, SVOP) which is to find $\bar{x} \in K$ such that for each $i \in I$,

$$\varphi_i(y) - \varphi_i(\bar{x}) \notin -\mathrm{int}C_i(\bar{x}) \quad \text{for all } y \in K,$$

where $\varphi_i(x) = (\varphi_{i_1}(x), \varphi_{i_2}(x), \ldots, \varphi_{i_{q_i}}(x))$ and for each $l \in \mathcal{L} = \{1, \ldots, q_i\}$, $\varphi_{i_l} : K \to \mathbb{R}$ is a function.

We can choose $y \in K$ in such a way that $y^i = \bar{x}^i$. Then the (SVOP) reduces to the *Nash equilibrium problem for vector-valued functions* which is to find $\bar{x} \in K$ such that for each $i \in I$,

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\mathrm{int}C_i(\bar{x}) \quad \text{for all } y_i \in K_i.$$

It is clear that every solution of the (SVOP) is also a solution of the Nash equilibrium problem for vector-valued functions, but the converse need not be true.

Now we recall some definitions.

DEFINITION 6. A real-valued function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *locally Lipschitz* if for any $z \in \mathbb{R}^p$ there exist a neighborhood $N(z)$ of z and a positive constant k such that

$$|f(x) - f(y)| \leqslant k||x - y|| \quad \text{for all } x, y \in N(z).$$

The Clarke *generalized directional derivative* (Clarke, 1990) of a locally Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ is

$$f^0(x; d) = \lim_{\substack{y \to x \\ t \downarrow 0}} \sup \frac{f(y + td) - f(y)}{t}.$$

The Clarke *generalized gradient* (Clarke, 1990)of a locally Lipschitz function f at x is defined as

$$\partial f(x) = \{\xi \in \mathbb{R}^p : f^0(x; d) \geqslant \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^p\}.$$

If f is convex, then the Clarke generalized gradient coincides with the subdifferential of f in the sense of convex analysis (Rockafellar, 1970).

The generalized invex function was introduced by Craven (1986) as a generalization of invex functions (Hanson, 1982).

DEFINITION 7. A locally Lipschitz function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *generalized invex at x w.r.t. a given function* $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ if

$$f(y) - f(x) \geqslant \langle \xi, \eta(y, x) \rangle \quad \text{for all } \xi \in \partial f(x) \text{ and } y \in \mathbb{R}^p.$$

For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be a locally Lipschitz function and let $x \in K$, $x_j \in K_j$. Following Clarke (1990), the *generalized directional derivative at x_j in the direction $d_j \in K_j$* of the function $\phi_i(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n)$ denoted by $\phi_{ij}^0(x; d_j)$ is

$$\phi_{ij}^0(x; d_j) = \lim_{\substack{y_j \to x_j \\ t \downarrow 0}} \sup \frac{1}{t}\{\phi_i(x_1, \dots, x_{j-1}, y_j + td_j, x_{j+1}, \dots, x_n)$$

$$-\phi_i(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n)\}.$$

The *partial generalized gradient* (Clarke, 1990) of the function $\phi_i(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$ at x_j is defined as follows:

$$\partial_j \phi_i(x) = \{\xi_j \in X_j : \phi_{ij}^0(x; d_j) \geqslant \langle \xi_j, d_j \rangle \text{ for all } d_j \in K_j\}.$$

LEMMA 2. (Clarke, 1990) *For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be locally Lipschitz. Then for each $i \in I$, the multivalued map $\partial_i \phi_i$ is upper semicontinuous.*

DEFINITION 8. For each $i \in I$, $\phi_i : K \to \mathbb{R}$ is called *generalized invex* at x w.r.t. a given function $\eta_i : K_i \times K_i \to \mathbb{R}^{p_i}$ if

$$\phi_i(y) - \phi_i(x) \geqslant \langle \xi_i, \eta_i(y_i, x_i) \rangle \quad \text{for all } \xi_i \in \partial_i \phi_i(x) \text{ and } y \in K.$$

PROPOSITION 1. *For each $i \in I$ and for all $l \in \mathcal{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$. Then any solution of the (SGVVLIP) is a solution of the (SVOP) with $T_i(x) = \partial_i \varphi_i(x)$ for each $i \in I$ and for all $x \in K$ where $\partial_i \varphi_i(x) = (\partial_i \varphi_{i_1}(x), \partial_i \varphi_{i_2}(x), \ldots, \partial_i \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{p_i \times q_i}$.*

Proof. For the sake of simplicity, we denote by $\varphi_i(x) = (\varphi_{i_1}(x), \ldots, \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{q_i}$, $u_i = (u_{i_1}, \ldots, u_{i_{q_i}})$ where $u_{i_l} \in \partial_i \varphi_{i_l}(x)$ for all $l \in \mathcal{L}$, and

$$\langle u_i, \eta_i(y_i, x_i) \rangle = (\langle u_{i_1}, \eta_{i_1}(y_i, x_i) \rangle, \ldots, \langle u_{i_{q_i}}, \eta_{i_{q_i}}(y_i, x_i) \rangle) \in \mathbb{R}^{q_i}.$$

Assume that $\bar{x} \in K$ is a solution of the (SGVVLIP). Then for each $i \in I$,

$$\forall y_i \in K_i, \ \ \exists \bar{u}_{i_l} \in \partial_i \varphi_{i_l}(\bar{x}) \text{ for all } l \in \mathcal{L} \text{ such that}$$

$$(\langle \bar{u}_{i_1}, \eta_{i_1}(y_i, \bar{x}_i) \rangle, \ldots, \langle \bar{u}_{i_{q_i}}, \eta_{i_{q_i}}(y_i, \bar{x}_i) \rangle) \notin -\text{int}C_i(\bar{x}).$$

We can rewrite this as

$$\forall y_i \in K_i, \ \ \exists \bar{u}_i \in \partial_i \varphi_i(\bar{x}) \ : \ \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int}C_i(\bar{x}). \tag{1}$$

Since for each $i \in I$ and for all $l \in \mathcal{L}$, φ_{i_l} is generalized invex w.r.t. η_{i_l}, we have

$$\varphi_{i_l}(y) - \varphi_{i_l}(\bar{x}) \geqslant \langle u_{i_l}, \eta_{i_l}(y_i, \bar{x}_i) \rangle \quad \text{for all } u_{i_l} \in \partial_i \varphi_{i_l}(\bar{x}) \text{ and } y \in K,$$

that is, for each $i \in I$

$$\varphi_i(y) - \varphi_i(\bar{x}) \geqslant \langle u_i, \eta_i(y_i, \bar{x}_i) \rangle \quad \text{for all } u_i \in \partial_i \varphi_i(\bar{x}) \text{ and } y \in K.$$

Therefore for each $i \in I$ and for all $u_i \in \partial_i \varphi_i(\bar{x})$, we have

$$\begin{aligned} \varphi_i(y) - \varphi_i(\bar{x}) &\in \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle + \mathbb{R}_+^{q_i} \\ &\subseteq \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle + \text{int}C_i(\bar{x}). \end{aligned} \tag{2}$$

From (1) and (2), we have $\varphi_i(y) - \varphi_i(\bar{x}) \notin -\text{int}C_i(\bar{x})$. Hence $\bar{x} \in K$ is a solution of the (SVOP).

THEOREM 4. *For each* $i \in I$ *and for all* $l \in \mathcal{L}$, *let* $\varphi_{i_l} : K \to \mathbb{R}$ *be generalized invex w.r.t.* $\eta_{i_l} : K_i \times K_i \to X_i$ *such that* η_{i_l} *is affine in the first argument, continuous in the second argument and* $\eta_{i_l}(x_i, x_i) = 0$ *for all* $x_i \in K_i$. *Assume that there exists* $r > 0$ *such that for all* $x \in K$, $||x|| > r$, *there exist* $i \in I$ *and* $\tilde{y}_i \in K_i$ *with* $||\tilde{y}_i||_i \leqslant r$ *satisfying*

$$\langle u_i, \eta_i(x_i, \tilde{y}_i)\rangle \in -\mathrm{int}C_i(x) \quad \textit{for all } u_i \in \partial_i\varphi_i(\bar{x}),$$

where $||\cdot||$ *and* $||\cdot||_i$ *denote the norms on* X *and* X_i, *respectively. Then the (SVOP) has a solution.*

Proof. For each $i \in I$ and for all $x \in K$, $T_i(x) = \partial_i\varphi_i(x)$ is an upper semicontinuous multivalued map by Lemma 2. It is easy to check that all conditions of Corollary 2 are satisfied. Hence from Corollary 2 and Proposition 1 it follows that the (SVOP) has a solution.

REMARK 4. (a) Theorem 4 and its specialization to lower semicontinuous convex functions improve Theorem 3.1 and Corollary 3.1 in Ansari and Yao (2000b), respectively, and extend these to vector-valued functions.
(b) Existence results for (SVOP) were also obtained in Ansari et al. (2000).

When the index set I is not necessarily finite, we can derive from Corollary 4 the following existence result for a solution of the Nash equilibrium problem for vector-valued functions.

THEOREM 5. *For each* $i \in I$, *let* K_i, X_i, Y_i, C_i *and* Q_i *be the same as in Theorem 3. For each* $i \in I$, *let* $\varphi_i : K \to Y_i$ *be a function which is* $C_i(x)$-*quasiconvex-like and continuous on* K. *For each* $i \in I$, *assume that there exist a nonempty compact subset* N *of* K *and a nonempty compact convex subset* B_i *of* K_i *for each* $i \in I$ *such that for each* $x \in K \setminus N$ *there exist* $i \in I$ *and* $\tilde{y}_i \in B_i$ *such that* $\varphi_i(x^i, \tilde{y}_i) - \varphi_i(x) \in -\mathrm{int}C_i(x)$. *Then the Nash equilibrium problem for vector-valued functions has a solution.*

Proof. For each $i \in I$, we set

$$F_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x) \quad \text{for all } x \in K,\ y_i \in K_i.$$

Then it is easy to verify the conditions of Corollary 4.

REMARK 5. Theorem 5 strengthens Corollary 3.17 in Tan and Tinh (1998) in several ways since our assumptions are weaker. Moreover it also generalizes Corollary 3.18 in Tan and Tinh (1998).

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